Mid-Cities Math Circle $(MC)^2$
Power of a Point
April 8, 2026

Question A. What necessary and sufficient conditions do we have for four points A, B, C, D to be concyclic (i.e. lie on a common circle)?

Question B. (Power of a Point Theorem) Let k be a fixed circle with center O and radius r, and P be a fixed point in the plane. A line l through P intersects k at A and B. Prove that the product $PA \cdot PB$ depends only on P and k, but not on the line l. Express $PA \cdot PB$ in terms of P and $k(O, r)$.

Remark. The product $PA \cdot PB$ can be understood as a signed product. What does that mean?

Definition. If P is a point, and $k(O, r)$ is a circle with center O and radius r in the plane, then $p(P, k) = OP^2 - r^2$ is called *the power of P with respect to k.*

Warm-up Problems

Problem 1. A circle passing through the vertices A, B of a triangle ABC intersects its sides AC, BC for the second time at L, K, respectively. Given that $AL = 2$, $LC = 6$, and $CK = 4$, find KB.

Problem 2. In a circle, chords AB and CD intersect at R. If $AR : BR = 1 : 4$ and $CR : DR = 4 : 9$, find the ratio $AB : CD$.

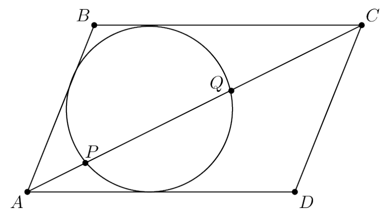
Problem 3. Let k_1 and k_2 be two intersecting circles. Let a common tangent to k_1 and k_2 touch k_1 at A_1 and k_2 at A_2. Show that the common chord of k_1 and k_2, when extended, bisects segment A_1A_2.

More Difficult Problems

Problem 4. Square $ABCD$ of side length 10 has a circle inscribed in it. Let M be the midpoint of AB. Find the length of that portion of the segment MC that lies outside of the circle.

Problem 5. Let BD be the angle bisector of angle B in triangle ABC with D on side AC. The circumcircle of triangle BDC meets AB at E, while the circumcircle of triangle ABD meets BC at F. Prove that $AE = CF$.

Problem 6. Let $ABCD$ be a parallelogram with $\angle BAD < 90°$. A circle tangent to sides $\overline{DA}$, $\overline{AB}$, and $\overline{BC}$ intersects diagonal $\overline{AC}$ at points P and Q with $AP < AQ$, as shown. Suppose that $AP = 3$, $PQ = 9$, and $QC = 16$. Find the area of $ABCD$.



Problem 7. Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. Find the area of the triangle ABC.

Problem 8. Let k_1 and k_2 be concentric circles, with k_2 in the interior of k_1. From a point A on k_1 one draws the tangent AB to k_2 ($B \in k_2$). Let C be the second point of intersection of AB and k_1, and let D be the midpoint of AB. A line passing through A intersects k_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio $AM : MC$.

Question C. Given two circles, one with center O_1 and radius r_1, the other with center O_2 and radius r_2, what is the set of points with equal power with respect to the two circles? Describe this set for all cases of the circles (intersecting, tangent, nonintersecting).
Answer: The *radical axis* of the two circles.

Problem 9. Let k_1, k_2, k_3 be three circles in the plane. Prove that the radical axes of k_1 and k_2, of k_2 and k_3, and k_1 and k_3, either all coincide, or are concurrent (or parallel).

Problem 10. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

Problem 11. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove the lines through A, B, C, perpendicular to the lines EF, FD, DE, respectively, are concurrent.

Problem 12. On the extension of chord KL of a circle centered at O, a point A is taken and tangents AP and AQ to the circle are drawn from it. Let M be the midpoint of PQ. Prove that $\angle MKO = \angle MLO$.

Problem 13. Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

Problem 14. A circle with center O passes through the vertices A and C of triangle ABC, and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangle ABC and KBN intersect at exactly two distinct points B and M. Prove that angle $\angle OMB$ is a right angle.