Mid-Cities Math Circle $(MC)^2$
Trigonometry
February 11, 2026

Warm-up Problems

Problem 1. Compute $\frac{\tan^2(10°)-\sin^2(10°)}{\tan^2(10°)\sin^2(10°)}$.

Problem 2. Compute $(\sin 15° + \sin 75°)^6$.

Problem 3. Evaluate:

$$(\tan 10°)(\tan 20°)(\tan 30°)(\tan 40°)(\tan 50°)(\tan 60°)(\tan 70°)(\tan 80°).$$

More Difficult Problems

Problem 4. If $\sum_{n=0}^{\infty} \cos^{2n}\theta = 5$, what is the value of $\cos 2\theta$?

Problem 5. Evaluate $(2-\sec^2 1°)(2-\sec^2 2°)(2-\sec^2 3°)\cdots(2-\sec^2 89°)$.

Problem 6. Find the positive integer n such that

$$\arctan\frac{1}{3} + \arctan\frac{1}{4} + \arctan\frac{1}{5} + \arctan\frac{1}{n} = \frac{\pi}{4}.$$

Problem 7. In triangle ABC, $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$. Then find $\angle C$ in degrees.

Problem 8. Solve in real numbers the equation $3x - x^3 = \sqrt{x+2}$.

Problem 9. Given that $(1+\tan 1°)(1+\tan 2°)\cdots(1+\tan 45°) = 2^n$, find n.

Problem 10. Given any 7 real numbers, prove that there are two of them x, y such that

$$0 \le \frac{x-y}{1+xy} \le \frac{1}{\sqrt{3}}.$$

Problem 11. The Chebyshev polynomials of the first kind can be defined by the recurrence relation $T_0(x) = 1$, $T_1(x) = x$, and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

(a) Prove that $T_n(\cos\theta) = \cos(n\theta)$ for all angles θ.

(b) Find all roots of T_n.

(c) Find the smallest positive integer n for which $\cos(18^\circ)$ is a root of $T_n(x)$: then find the exact value of $\sin(18^\circ)$.

Problem 12. Prove that for all positive real numbers a, b, c we have

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

Problem 13. Prove that if α is a real number such that

$$\cos\pi\alpha = 1/3,$$

then α is irrational. (The angle $\pi\alpha$ is in radians.)