Introduction to Probability

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Mid Cities Math Circle

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Problems

1. Dimitar randomly choose a positive integers between 1 and 50. What is the probability that the number
   (a) is odd?
   (b) has a unit digit 4?
   (c) is a prime number?
   (d) is a perfect square?
   (e) is a divisors of 45?
   (f) is a multiple of 3?
   (g) is a multiple of 3 but not a multiple of 6.?
   (h) is a multiple of 2 or 3?
   (i) is a multiple of 2 or 3, but not a multiple of 6?
2. Nikolay draw a card from a standard deck of 52 cards. What is the probability that
   (a) the card is a 2?
   (b) the card is a King of Diamond?
   (c) the card is a face card?
   (d) the card is an Ace or Club?
3. What is the probability that a randomly drawn positive factor of 60 is less than 7?
4. A complete cycle of a traffic light takes 60 seconds. During each cycle the light is green for 25 seconds, yellow for 5 seconds, and red for 30 seconds. At a randomly chosen time, what is the probability that the light will NOT be green?
5. A box contains 5 blue balls and 7 reds ball.
   (a) A ball is drawn out of the box at random. What is the probability that the ball is red?
   (b) Two balls are drawn out of the box at random. What is the probability that both balls are red?
   (c) Two balls are drawn out of the box at random. What is the probability that both balls are white?
   (d) Two balls are drawn out of the box at random. What is the probability that both balls have the same color?
6. One fair die has faces 1, 1, 2, 2, 3, 3 and another has faces 4, 4, 5, 5, 6, 6. The dice are rolled and the numbers on the top faces are added. What is the probability that the sum will be odd?

7. A pair of six-sided dice are labeled so that one die has only even numbers (two each of 2, 4, and 6), and the other die has only odd numbers (two of each 1, 3, and 5). The pair of dice is rolled. What is the probability that the sum of the numbers on the tops of the two dice is 7?

8. Two eight-sided dice each have faces numbered 1 through 8. When the dice are rolled, each face has an equal probability of appearing on the top. What is the probability that the product of the two top numbers is greater than their sum?

9. Twelve fair dice are rolled. What is the probability that the product of the numbers on the top faces is prime?

10. Two fair coins are to be tossed once. For each head that results, one fair die is to be rolled. What is the probability that the sum of the die rolls is odd? (Note that if no die is rolled, the sum is 0.)

11. the numbers $1, 2, \ldots, 10$ are arranged in a row at random. Find the probability that the 2 comes immediately after the 1.

12. A wooden cube is painted red on all sides. It is then cut into 1000 smaller cubes, all the same size. One of the small cubes is chosen at random, and then tossed. The probability that a red face appears on the top can be expressed in the from m/n, where m and n are relatively prime positive integers. Find m+n.

13. Two real numbers are selected independently at random from the interval $[-20, 10]$. What is the probability that the product of those numbers is greater than zero?

14. The numbers x and y are chosen at random in the interval $[0, 1]$. Find the probability that $1, x, y$ form the sides of an obtuse triangle.

15. A diameter 1ft, frisbee disc. landed on a 10×6 rectangular table. What is the probability that the disc lie entirely on the table?

16. A point (x, y) is randomly picked from inside the rectangle with vertices (0, 0), (4, 0), (4, 1), and (0, 1). What is the probability that $x < y$?

17. A points randomly choosing from inside of an equilateral triangle ABC. What is the probability that the point is closer to vertex A than other vertices?

18. An unfair coin is flipped twice. The probability of getting two Heads is 0.81. Compute the probability flipping the coin twice and getting two Tails. (ARML Local 2008)

19. Jacob flips five coins, exactly three of which land heads. What is the probability that the first two are both heads? (HMMT 2010)

20. A jar has 10 red candies and 10 blue candies. Terry picks two candies at random, then Mary picks two of the remaining candies at random. Given that the probability that they get the same color combination, irrespective of order, is m/n, where m and n are relatively prime positive integers, find $m + n$. (AIME 2004)

21. A hotel packed breakfast for each of three guests. Each breakfast should have consisted of three types of rolls, one each of nut, cheese, and fruit rolls. The preparer wrapped each of the nine rolls and once wrapped, the rolls were indistinguishable from one another. She then randomly put three rolls in a bag for each of the guests. Given that the probability each guest got one roll of each type is $\frac{m}{n}$, where m and n are relatively prime integers, find $m + n$. (AIME 2005)

22. Let A, B, and C be randomly chosen (not necessarily distinct) integers between 0 and 4 inclusive. Pat and Chris compute the value of $A + B \cdot C$ by two different methods. Pat follows the proper order of operations, computing $A + (B \cdot C)$. Chris ignores order of operations, choosing instead to compute $(A + B) \cdot C$. Compute the probability that Pat and Chris get the same answer. (ARML 2014)

23. There is a 40% chance of rain on Saturday and a 30% chance of rain on Sunday. However, it is twice as likely to rain on Sunday if it rains on Saturday than if it does not rain on Saturday. The probability that it rains at least one day this weekend is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$. (AIME 2016)

24. Twenty-seven players are randomly split into three teams of nine. Given that Zack is on a different team from Mihir and Mihir is on a different team from Andrew, what is the probability that Zack and Andrew are on the same team? (HMMT 2018)

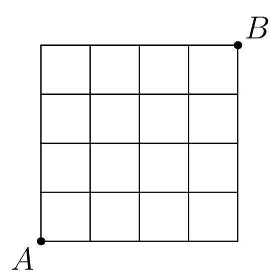
25. Lines l_1 and l_2 are represented by the equations $y = m_1x + 10$ and $y = m_2x + 9$, respectively. If m_1 is randomly selected from $\left\{0!, \left(\frac{1}{2}\right)^{-1}, \sqrt[3]{-1}\right\}$ and m_2 is randomly selected from $\left\{\frac{6!}{5!}, -\sqrt{\frac{1}{4}}, -1\right\}$, what is the probability that l_1 and l_2 are parallel? (MathCount 1994)

26. Charles has two six-sided die. One of the die is fair, and the other die is biased so that it comes up six with probability $\frac{2}{3}$ and each of the other five sides has probability $\frac{1}{15}$. Charles chooses one of the two dice at random and rolls it three times. Given that the first two rolls are both sixes, the probability that the third roll will also be a six is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$. (AIME 2014)

27. The probability that 11 consecutive throws of a pair of ordinary fair dice will each have a different total can be written in the form $\frac{(a!)(b!)(c!)}{36^d}$ where a, b, c, and d are integers and $a \le b \le c \le d$. Compute the ordered quadruple (a, b, c, d). (ARML 1993)

28. George has two coins, one of which is fair and the other of which always comes up heads. Jacob takes one of them at random and flips it twice. Given that it came up heads both times, what is the probability that it is the coin that always comes up heads? (HMMT 2010)

29. On the grid shown, A and B begin at the points indicated. Let each small segment of the grid be thought of as one block. A and B begin moving at the same time, each moving at the rate of one block per minute. A may only move to the right or upward (with equal probability, when there is a choice); B may only move to the left or downward (with equal probability, when there is a choice). Each travels to the others' starting position (thus each makes a trip of 8 blocks). Compute the probability that they will meet at some point during their trip. (ARML 1983)



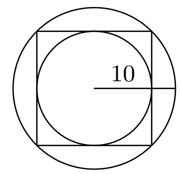
30. Two numbers choosing randomly from the interval $[0, 1]$. What is the probability that

    (a) $x + y > \frac{3}{2}$?
    (b) $y^2 + x^2 > \frac{1}{2}$?
    (c) $\frac{y}{x} > 10$

31. A point (x, y) is randomly selected such that $0 \le x \le 8$ and $0 \le y \le 4$. What is the probability that both $x \le 2$ and $y \le 2$? Express your answer as a common fraction.

    [2001 MATHCOUNTS Chapter Team #1]

32. A dart is thrown at the dart board shown and is equally likely to land anywhere inside the large circle of radius 10. What is the probability that it will land inside the square but not inside the inner circle? Express your answer rounded to the nearest whole percent.



[1995 MATHCOUNTS National Team #1]

33. Darts are thrown randomly at a board showing three concentric circles of radii 2, 3, and 4. If a dart hits inside one of the circles, what is the probability, expressed as a common fraction, that it is in the interior of the circle of radius 3 but not in the interior of the circle of radius 2?

    [1994 MATHCOUNTS National Sprint #2]

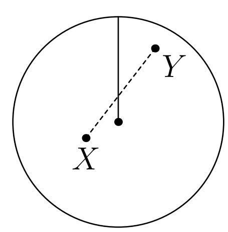
34. A point P is chosen at random in the interior of a unit square S. Let $d(P)$ denote the distance from P to the closest side of S. The probability that $\frac{1}{5} \le d(P) \le \frac{1}{3}$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

    [2010 AIME II #2]

35. Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After waiting at that gate, Dave is told the departure gate has been changed to a different gate, again at random. Let the probability that Dave walks 400 feet or less to the new gate be a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

    [2010 AIME II #4]

36. A circle is drawn, along with a particular radius. Next two points X and Y are chosen independently at random inside the circle. What is the probability that segment $\overline{XY}$ intersects the given radius, as shown?



    [2019 Mandelbrot Round 1 #5]

37. A point is selected at random from the interior of a right triangle with legs of length $2\sqrt{3}$ and 4. Let p be the probability that the distance between the point and the nearest vertex is less than 2. Then p can be written in the form $a + \sqrt{b}\pi$, where a and b are rational numbers. Compute (a, b).

    [2014 ARML T-2]

38. Laurel appears at the Comedy Cafe at a random time between 1:00 and 2:00. Hardy turns up at a random time between 2:00 and 2:30. What is the probability that Laurel waits for no more than 40 minutes for Hardy?

    [2000 Mandelbrot Round 3 #6]

39. The train schedule in Hummut is hopelessly unreliable. Train A will enter Intersection X from the west at a random time between 9:00 am and 2:30 pm; each moment in that interval is equally likely. Train B will enter the same intersection from the north at a random time between 9:30 am and 12:30 pm, independent of Train A; again, each moment in the interval is equally likely. If each train takes 45 minutes to clear the intersection, what is the probability of a collision today?

    [2006 HMMT-F General Part 1 #7]

40. Chloe chooses a real number uniformly at random from the interval $[0, 2017]$. Separately, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chloe's number?

[2017 AMC 12A #10]

(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{5}{6}$ **(E)** $\frac{7}{8}$

41. A traffic light runs repeatedly through the following cycle: green for 30 seconds, then yellow for 3 seconds, and then red for 30 seconds. Leah picks a random three-second time interval to watch the light. What is the probability that the color changes while she is watching?

[2007 AMC 12B #13]

(A) $\frac{1}{63}$ **(B)** $\frac{1}{21}$ **(C)** $\frac{1}{10}$ **(D)** $\frac{1}{7}$ **(E)** $\frac{1}{3}$

42. A point P is chosen at random in the interior of equilateral triangle ABC. What is the probability that $\triangle ABP$ has a greater area than each of $\triangle ACP$ and $\triangle BCP$?

[2003 AMC 12A #16]

(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

43. Let $ABCD$ be a unit square. Points E and F are chosen randomly, uniformly, and independently on $\overline{AB}$ and $\overline{CD}$, respectively, with all points on each segment equally likely to be chosen. Compute the probability that

$$\max\{[AEFD], [BEFC]\} > \frac{3}{4}.$$

[2020 ARML Local I-7]

44. A point P is randomly selected from the rectangular region with vertices $(0, 0), (2, 0), (2, 1), (0, 1)$. What is the probability that P is closer to the origin than it is to the point $(3, 1)$?

[2002 AMC 12B #18]

(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{4}{5}$ **(E)** 1

45. Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the start line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture?

[2009 AMC 12B #18]

(A) $\frac{1}{16}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{3}{16}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{5}{16}$

46. A frog makes 3 jumps, each exactly 1 meter long. The directions of the jumps are chosen independenly at random. What is the probability that the frog's final position is no more than 1 meter from its starting position?

[2010 AMC 12B #18]

(A) $\dfrac{1}{6}$ **(B)** $\dfrac{1}{5}$ **(C)** $\dfrac{1}{4}$ **(D)** $\dfrac{1}{3}$ **(E)** $\dfrac{1}{2}$

47. Cyrus the frog sits on a flat surface. He jumps, landing 2 feet away. He then chooses a direction at random and again jumps 2 feet. What is the probability that after the second jump Cyrus lands within 1 foot of his starting position?

[2023 AMC 12B #20]

(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{\sqrt{3}}{8}$ **(D)** $\frac{\arctan\frac{1}{2}}{\pi}$ **(E)** $\frac{2\arcsin\frac{1}{4}}{\pi}$

48. A circle of radius 1 is randomly placed in a 15-by-36 rectangle $ABCD$ so that the circle lies completely within the rectangle. Given that the probability that the circle will not touch diagonal AC is m/n, where m and n are relatively prime positive integers. Find $m+n$.

    [2004 AIME I #10]

49. Real numbers x, y, and z are chosen at random from the unit interval $[0,1]$. Compute the probability that $\max\{x,y,z\} - \min\{x,y,z\} \le \frac{2}{3}$.

    [2016 ARML T-7]

50. Two circles of radius 1 are to be constructed as follows. The center of circle A is chosen uniformly and at random from the line segment joining $(0,0)$ and $(2,0)$. The center of circle B is chosen uniformly and at random, and independently of the first choice, from the line segment joining $(0,1)$ to $(2,1)$. What is the probability that circles A and B intersect?

    [2008 AMC 12B #21]

    (A) $\frac{2+\sqrt{2}}{4}$ **(B)** $\frac{3\sqrt{3}+2}{8}$ **(C)** $\frac{2\sqrt{2}-1}{2}$ **(D)** $\frac{2+\sqrt{3}}{4}$
    (E) $\frac{4\sqrt{3}-3}{4}$

51. Let S be a square of side length 1. Two points are chosen independently at random on the sides of S. The probability that the straight-line distance between the points is at least $\frac{1}{2}$ is $\frac{a-b\pi}{c}$, where a, b, and c are positive integers and $\gcd(a,b,c)=1$. What is $a+b+c$?

    [2015 AMC 12A #23]

    (A) 59 **(B)** 60 **(C)** 61 **(D)** 62 **(E)** 63

52. Three points are chosen randomly and independently on a circle. What is the probability that all three pairwise distances between the points are less than the radius of the circle?

    [2003 AMC 12B #25]

    (A) $\frac{1}{36}$ **(B)** $\frac{1}{24}$ **(C)** $\frac{1}{18}$ **(D)** $\frac{1}{12}$ **(E)** $\frac{1}{9}$

53. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\dfrac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

54. Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r+s\pi$, where r and s are rational numbers.

    [1993 Putnam B3]

55. If three points are selected at random on a given circle, find the probability that these three points lie on a semicircle.

56. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

57. Let k be a positive integer. Suppose that the integers $1,2,3,\ldots,3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

58. (Hard) Three points are selected randomly on the circumference of a circle. What is the probability that the triangle formed by there three points contains the center of a circle.

59. (Very Hard) Suppose we have a floor made of parallel strips of wood, each the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips?

60. A permutation $\{x_1, x_2, \ldots, x_{2n}\}$ of the set $\{1, 2, \ldots, 2n\}$ where n is a positive integer, is said to have property T if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that, for each n, there are more permutations with property T than without. (**Note:** Someone in the audience did try to solve this during the contest).