Mid-Cities Math Circle $(MC)^2$
Complex Numbers
April 30, 2025

1. We define i to be a number such that $i^2 = -1$. We define a complex number z to be a number of the form $a + bi$ where a and b are real numbers. We denote the conjugate of $z = a + bi$ with $\overline{z} = a - bi$.

2. Addition and subtraction over complex numbers are defined componentwise, i.e. $(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$.

3. Multiplication over complex numbers is defined according to the distributive law, i.e. $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

4. To every complex number $z = a + bi$ we assign the point (a, b) in the coordinate plane. In particular, to 0 we assign the origin $(0, 0)$.

5. The **norm** $|z|$ is defined to be $\sqrt{a^2 + b^2}$, which is the distance from 0 to z.

6. The **argument** $\arg z$ is defined to be the counterclockwise angle from the positive real axis to the line connecting 0 and z. Note that the argument of z is not defined uniquely but it is unique "modulo 2π" (The argument of 0 is not defined.)

7. Every nonzero complex number z can be expressed uniquely as: $z = r(\cos\theta + i\sin\theta)$ where r is a positive real number and $0 \leq \theta < 2\pi$. ($r = |z|$ and $\theta = \operatorname{Arg} z$). We have that $z^n = r^n(\cos n\theta + i\sin n\theta)$

Warm-up Problems

Problem 1. Find $(1 + i)^{20} - (1 - i)^{20}$.

Problem 2. Compute $(1 + i)^{2025}$.

Problem 3. The complex number z is equal to $9 + bi$, where b is a positive real number. Given that the imaginary parts of z^2 and z^3 are the same, what is b equal to?

More Difficult Problems

Problem 4. Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3°$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

Problem 5. There is a unique pair of positive real numbers satisfying the equations

$$x^4 - 6x^2y^2 + y^4 = 8 \quad \text{and} \quad x^3y - xy^3 = 2\sqrt{3}.$$

Determine x, writing your answer in the form $a\cos\theta$, with θ in degrees.

Problem 6. Let (x, y) be a pair of real numbers satisfying

$$56x + 33y = \frac{-y}{x^2+y^2}, \qquad \text{and} \qquad 33x - 56y = \frac{x}{x^2+y^2}.$$

Determine the value of $|x| + |y|$.

Problem 7. Suppose that x, y, and z are complex numbers such that $xy = -80 - 320i$, $yz = 60$, and $zx = -96 + 24i$, where $i = \sqrt{-1}$. Then there are real numbers a and b such that $x + y + z = a + bi$. Find $a^2 + b^2$.

Problem 8. Let z be a complex number satisfying $12|z|^2 = 2|z+2|^2 + |z^2+1|^2 + 31$. What is the value of $z + \frac{6}{z}$?

Problem 9. There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

Find n.

Problem 10. Suppose that x, y, and z are complex numbers of equal magnitude that satisfy

$$x + y + z = -\frac{\sqrt{3}}{2} - i\sqrt{5}$$

and

$$xyz = \sqrt{3} + i\sqrt{5}.$$

If $x = x_1 + ix_2$, $y = y_1 + iy_2$, and $z = z_1 + iz_2$ for real x_1, x_2, y_1, y_2, z_1, and z_2, then

$$(x_1x_2 + y_1y_2 + z_1z_2)^2$$

can be written as a/b for relatively prime positive integers a and b. Compute $100a + b$.

Problem 11. Let $P(z) = z^n + c_1z^{n-1} + c_2z^{n-2} + \cdots + c_n$ be a polynomial in the complex variable z, with real coefficients c_k. Suppose that $|P(i)| < 1$. Prove that there exist real numbers a and b such that $P(a + bi) = 0$ and $(a^2 + b^2 + 1)^2 < 4b^2 + 1$.