Mid-Cities Math Circle $(MC)^2$
Factoring and Partial Fractions
March 5, 2025

Warm-up Problems

Problem 1. How many positive integer factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely $1, 2, 3, 4, 6$, and 12.)
Problem 2. Let $t_n = \frac{n(n+1)}{2}$ be the nth triangular number. Find

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + ... + \frac{1}{t_{2025}}$$

Problem 3. Find he sum

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{2024}{2025!}$$

More Difficult Problems

Problem 4. Find the sum $\frac{1}{1(3)} + \frac{1}{3(5)} + \cdots + \frac{1}{(2n-1)(2n+1)} + \cdots + \frac{1}{2023(2025)}$.

Problem 5. Consider the sequence defined by $a_k = \frac{1}{k^2+k}$ for $k \geq 1$. Given that $a_m + a_{m+1} + \cdots + a_{n-1} = 1/29$, for positive integers m and n with $m < n$, find $m + n$.

Problem 6. Evaluate the expression

$$\frac{121\left(\frac{1}{13} - \frac{1}{17}\right) + 169\left(\frac{1}{17} - \frac{1}{11}\right) + 289\left(\frac{1}{11} - \frac{1}{13}\right)}{11\left(\frac{1}{13} - \frac{1}{17}\right) + 13\left(\frac{1}{17} - \frac{1}{11}\right) + 17\left(\frac{1}{11} - \frac{1}{13}\right)}.$$

Problem 7. Compute

$$\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}.$$

Problem 8. Determine the value of the sum

$$\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots + \frac{29}{14^2 \cdot 15^2}.$$

Problem 9. Given:

$$S = 1 + \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}+\frac{1}{6}} + \cdots + \frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}}$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers (i.e. $k = \frac{n(n+1)}{2}$ for $n = 1,\ 2,\ \ldots,1996$). Prove that $S > 1001$.

Problem 10. Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

Problem 11. Find $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n}$.

Problem 12. For $n \geq 3$, let $f(n)$ be the number of subsets of three elements that can be chosen from a set of n distinct elements. Compute

$$\sum_{n=3}^{101} \frac{1}{f(n)}.$$