Linear Algebra and Combinatorics, at the UTA $(MC)^2$

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Problems:

Problem 1. Assume that $x_1, x_2, \dots, x_7$ are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Determine

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

Problem 2. Let M_n be the $n \times n$ matrix with entries as follows: for $1 \leq i \leq n$, $m_{i,i} = 10$; for $1 \leq i \leq n-1, m_{i+1,i} = m_{i,i+1} = 3$; all other entries in M_n are zero. Let D_n be the determinant of matrix M_n. Determine $\displaystyle\sum_{n=1}^{\infty} \frac{1}{8D_n + 1}$.

Note: The determinant of the 1×1 matrix $[a]$ is a, and the determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$; for $n \geq 2$, the determinant of an $n \times n$ matrix with first row or first column $a_1\ a_2\ a_3 \dots\ a_n$ is equal to $a_1C_1 - a_2C_2 + a_3C_3 - \dots + (-1)^{n+1}a_nC_n$, where C_i is the determinant of the $(n-1) \times (n-1)$ matrix found by eliminating the row and column containing a_i.

Problem 3. Let M_n denote the $(n-1) \times (n-1)$ matrix

$$\begin{bmatrix} 3 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 5 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & n+1 \end{bmatrix}.$$

Let $D_n = \det(M_n)$. Is the set $\left\{ \frac{D_n}{n!} | n \geq 2 \right\}$ bounded?

Problem 4. Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is $\frac{1}{\min(i,j)}$ for $1 \leq i, j \leq n$. Compute $\det(A)$.

Problem 5. Do there exist square matrices A and B of the same size such that $AB - BA$ is the identity matrix?

Problem 6. Do there exist square matrices A and B of the same size such that $ABAB = 0$ and $BABA \neq 0$?

Problem 7. Let n and k be positive integers. Say that a permutation σ of $\{1, 2, \dots n\}$ is k-limited if $|\sigma(i) - i| \leq k$ for all i. Prove that the number of k-limited permutations of $\{1, 2, \dots n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.

Problem 8. Let n be a positive integer. Suppose that $A_1, \dots, A_{n+1}$ are subsets of $\{1, \dots, n\}$. Suppose that $|A_i|$ is odd for all i. Prove that there exist i and j with $i \neq j$ such that $|A_i \cap A_j|$ is odd.

Problem 9. Let n be an even positive integer. Suppose that $A_1, \dots, A_n$ are subsets of $\{1, \dots, n\}$. Suppose that $|A_i|$ is even for all i. Prove that there exist i and j with $i \neq j$ such that $|A_i \cap A_j|$ is even.

Problem 10. Let n be a positive integer. Suppose that $A_1, \dots, A_m$ are distinct subsets of $\{1, \dots, n\}$. Suppose that $|A_i|$ is even for all i. Suppose that there do not exist i and j with $i \neq j$ such that $|A_i \cap A_j|$ is odd. Determine the largest possible value of m, in terms of n.

Problem 11. Let S be a finite set. Suppose that there exists a subset T of $S \times S$ with the following property: for all $a, b \in S$, there exists exactly one element c of S with the property that $(a, c), (c, b) \in S$. Determine all possible values of $|S|$.

More Problems:

Problem 12. Suppose S is a finite nonempty set of invertible $n \times n$ matrices that is closed under multiplication (so $A, B \in S \implies AB \in S$). Let M be the sum of the elements of S. Given that the trace of M is 0, prove that $M = 0$. (The *trace* of a matrix is the sum of the elements on its main diagonal [the diagonal from upper-left to lower-right].)

Problem 13. Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the kth power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \cdots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.

Problem 14. Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.

Problem 15. Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m+n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

Problem 16. Suppose we have a list of tuples $(A_1, b_1), \cdots, (A_n, b_n)$, where each A_i is an $n \times n$ strictly lower triangular matrix over $\mathbb{F}_2$ and each b_i is a column vector in $\mathbb{F}_2^n$. Suppose we can perform the following operations:

1. "Swap": swap the indices of two pairs (A_i, b_i) and (A_j, b_j), or
2. "Compose": choose two distinct indices i and j, and update A_i to be $A_i + A_j + A_i A_j$ and update b_i to be $b_i + b_j + A_i b_j$.

Then prove that it is possible to perform operations either to reach a state where $b_i = e_i$ for all i, or to reach a state where some b_i is 0. (Recall that e_i denote the standard basis vectors of $\mathbb{F}_2^n$.)

Problem 17. Let n be a positive integer. Let $A_1, \ldots, A_{n+1}$ be nonempty subsets of $\{1, \ldots, n\}$. Prove that there exist disjoint nonempty subsets I and J of $\{1, \ldots, n+1\}$ such that $\cup_{i \in I} A_i = \cup_{j \in J} A_j$.

Problem 18. At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is of the form 2^k for some positive integer k.

Problem 19. Let n be a positive integer. What is the largest k for which there exist $n \times n$ matrices $M_1, \ldots, M_k$ and $N_1, \ldots, N_k$ with real entries such that for all i and j, the matrix product $M_i N_j$ has a zero entry somewhere on its diagonal if and only if $i \neq j$?

Problem 20. Let $C_m = \frac{1}{m+1}\binom{2m}{m}$ be the mth Catalan number. Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is C_{i+j-1} for $1 \leq i, j \leq n$. Compute $\det(A)$.

Problem 21.
(a) Let G be a graph on n vertices. Suppose G is a tree (a connected graph with no cycles). Real numbers $x_1, \ldots, x_n$ are written on the vertices of G (with one number per vertex). For each edge, the product of the numbers on the two vertices of the edge is calculated. Let S be the sum, over all edges, of this calculated product. Prove that

$$\sqrt{n-1}(x_1^2 + x_2^2 + \cdots + x_n^2) \geq 2S.$$

(b) The same problem, except instead of assuming G is a tree, we only assume that G has no cycles of length less than 5.

Problem 22.
(a) Let k, ℓ, t be positive integers. Suppose that $A_1, \ldots, A_t$ and $B_1, \ldots, B_t$ are sets such that

- for all i, $|A_i| \leq k$ and $|B_i| \leq \ell$;
- for all i and j with $i \neq j$, $A_i \cap B_j \neq \emptyset$.

Prove that $t \leq \binom{k+\ell}{k}$.
(b) The same problem, except, in the last bullet point, change "$i \neq j$" to "$i < j$".