Mid-Cities Math Circle $(MC)^2$
Quadrilaterals and Ptolemy's Theorem
February 12, 2025

We say that $ABCD$ is *cyclic* if A, B, C, D lie on a circle in that order.

Theorem. Let the diagonals AC and BD of $ABCD$ intersect at X. The quadrilateral $ABCD$ is cyclic if and only if any of the following equivalent conditions hold:

(i) $\angle ABD = \angle ACD$.

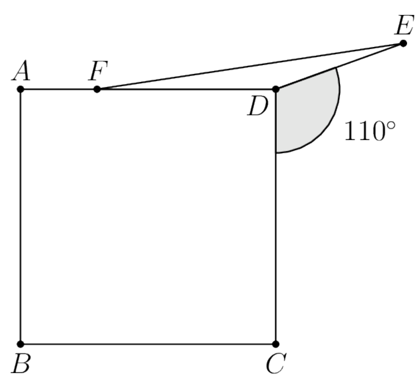
(ii) Opposite angles sum to $180°$.

(iii) $AX \cdot XC = BX \cdot XD$.

(iv) $AB \cdot CD + BC \cdot AD = AC \cdot BD$.

Warm-up Problems

Problem 1. As shown in the figure below, point E lies on the opposite half-plane determined by line CD from point A so that $\angle CDE = 110°$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



Problem 2. A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

Problem 3. In triangle ABC we have $AB = 7$, $AC = 8$, $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of $\frac{AD}{CD}$?

More Difficult Problems

Problem 4. Let P be a point on the arc BC of the circumcircle of an equilateral triangle ABC that does not contain the vertex A. Prove that $PA = PB + PC$.

Problem 5. In a regular heptagon $ABCDEFG$, prove that $\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}$.

Problem 6. A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.

Problem 7. Let P be a point on the circle circumscribing square $ABCD$ that satisfies $PA \cdot PC = 56$ and $PB \cdot PD = 90$. Find the area of $ABCD$.

Problem 8. In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

Problem 9. Let $ABCDE$ be a pentagon inscribed in a circle such that $AB = CD = 3$, $BC = DE = 10$, and $AE = 14$. Find the sum of the lengths of all diagonals of $ABCDE$.

Problem 10. Let $A_1A_2A_3A_4A_5A_6$ be a hexagon inscribed inside a circle of radius r. Furthermore, for each positive integer $1 \leq i \leq 6$ let M_i be the midpoint of the segment $\overline{A_iA_{i+1}}$, where $A_7 \equiv A_1$. Suppose that hexagon $M_1M_2M_3M_4M_5M_6$ can also be inscribed inside a circle. If $A_1A_2 = A_3A_4 = 5$ and $A_5A_6 = 23$, find r.

Problem 11. Let $a > b > c > d$ be positive integers and suppose that $ac + bd = (b + d + a - c)(b + d - a + c)$. Prove that $ab + cd$ is not prime.

Problem 12. Prove that $\csc \frac{\pi}{7} = \csc \frac{2\pi}{7} + \csc \frac{3\pi}{7}$ using Ptolemy's Theorem.

Problem 13. A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.