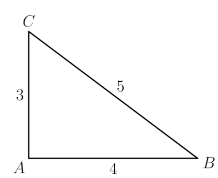
Mid-Cities Math Circle $(MC)^2$
Selected Problems in Triangle Geometry
October 16, 2024

We will use the following notation for a generic triangle ABC: $BC = a$, $CA = b$, and $AB = c$; the angle $\angle BAC$ will be shortly denoted by $\angle A$, etc.

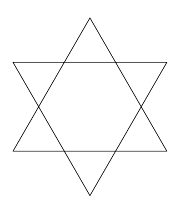
- *Law of Cosines:*
$$a^2 = b^2 + c^2 - 2bc \cdot \cos\angle A.$$
- *Extended Law of Sines:*
$$\frac{a}{\sin\angle A} = 2R,$$
where R is the circumradius of $\triangle ABC$.
- *Angle Bisector Theorem:* If the angle bisector of $\angle A$ of the triangle ABC intersects the opposite side BC at D, then
$$\frac{DB}{DC} = \frac{AB}{AC}.$$

Warm-up Problems

Problem 1. In the figure below, choose point D on $\overline{BC}$ so that $\triangle ACD$ and $\triangle ABD$ have equal perimeters. What is the area of $\triangle ABD$?



Problem 2. A unit hexagram is composed of a regular hexagon of side length 1 and its 6 equilateral triangular extensions, as shown in the diagram. What is the ratio of the area of the extensions to the area of the original hexagon?



Problem 3. Two angles of an isosceles triangle measure $70°$ and $x°$. What is the sum of the three possible values of x?

More Difficult Problems

Problem 4. Triangle ABC has a right angle at B and contains a point P such that $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC.

Problem 5. In quadrilateral $ABCD$, $BC = 4$, $CD = 7$, $AD = 1$, and $\angle A = \angle B = 60°$. Find the distance AB.

Problem 6. Let $ABCD$ be a rhombus with $\angle ADC = 46°$. Let E be the midpoint of $\overline{CD}$, and let F be the point on $\overline{BE}$ such that $\overline{AF}$ is perpendicular to $\overline{BE}$. What is the degree measure of $\angle BFC$?

Problem 7. Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1$, $BP = \sqrt{3}$, and $CP = 2$. What is s?

Problem 8. Let $\triangle ABC$ be a scalene triangle. Point P lies on $\overline{BC}$ so that $\overline{AP}$ bisects $\angle BAC$. The line through B perpendicular to $\overline{AP}$ intersects the line through A parallel to $\overline{BC}$ at point D. Suppose $BP = 2$ and $PC = 3$. What is AD?

Problem 9. In rectangle $ABCD$, $\overline{AB} = 20$ and $\overline{BC} = 10$. Let E be a point on $\overline{CD}$ such that $\angle CBE = 15°$. What is $\overline{AE}$?

Problem 10. In an equilateral triangle ABC, there lies a point P in the interior such that $AP^2 = BP^2 + CP^2$. Find $\angle BPC$.

Problem 11. Triangle ABC is isosceles with $AC = BC$ and $\angle ACB = 106°$. Point M is in the interior of the triangle so that $\angle MAC = 7°$ and $\angle MCA = 23°$. Find the number of degrees in $\angle CMB$.

Problem 12. Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.