Mid-Cities Math Circle $(MC)^2$
Inequalities
October 30, 2024

The *Cauchy-Schwarz inequality* for (a, b, c) and (x, y, x) states that

$$(x^2 + y^2 + z^2)(a^2 + b^2 + c^2) \geq (ax + by + cz)^2,$$

with equality if and only if there exists a scalar k for which $x : a = y : b = z : c = 1 : k$ (provided that not all x, y, z are zero).

The *AM–GM inequality* for n nonnegative variables $a_1, a_2, \ldots, a_n$ states that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Warm-up Problems

Problem 1. Which is larger: (a) $\frac{99999}{100000}$ or (b) $\frac{100000}{100001}$?

Problem 2. What is the smallest whole number larger than the perimeter of any triangle with a side of length 5 and a side of length 19?

Problem 3. Fifteen integers $a_1, a_2, a_3, \ldots, a_{15}$ are arranged in order on a number line. The integers are equally spaced and have the property that

$$1 \leq a_1 \leq 10,\ 13 \leq a_2 \leq 20, \text{ and } 241 \leq a_{15} \leq 250.$$

What is the sum of digits of a_{14}?

More Difficult Problems

Problem 4. Let x, y and z be positive real numbers.

(a) If $x + y + z \geq 3$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 3$?

(b) If $x + y + z \leq 3$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3$?

Problem 5. Find the smallest integer n such that $(x^2 + y^2 + z^2)^2 \leq n(x^4 + y^4 + z^4)$ for all real numbers x, y, and z.

Problem 6. If a, b, c, x, y, and z are real and $a^2+b^2+c^2 = 25$, $x^2+y^2+z^2 = 36$, and $ax + by + cz = 30$, compute $\frac{a+b+c}{x+y+z}$.

Problem 7. Let x and y be two real numbers such that $2\sin x \sin y+3\cos y+6\cos x \sin y = 7$. Find $\tan^2 x + 2\tan^2 y$.

Problem 8. Show that, for all positive real numbers p, q, r, and s,

$$(p^2 + p + 1)(q^2 + q + 1)(r^2 + r + 1)(s^2 + s + 1) \geq 81pqrs.$$

Problem 9. Prove that

$$1 - \frac{1}{2012}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2013}\right) \geq \frac{1}{\sqrt[2012]{2013}}.$$

Problem 10. Determine the maximum value attained by

$$\frac{x^4 - x^2}{x^6 + 2x^3 - 1}$$

over real numbers $x > 1$.

Problem 11. Suppose that $a, b, c > 0$ such that $abc = 1$. Prove that

$$\frac{ab}{ab + a^5 + b^5} + \frac{bc}{bc + b^5 + c^5} + \frac{ca}{ca + c^5 + a^5} \leq 1.$$

Problem 12. Let a, b, and c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$