Mid-Cities Math Circle $(MC)^2$
Polynomials
September 18, 2024

A polynomial is a function in one or more variables that consists of a sum of variables raised to nonnegative, integral powers and multiplied by coefficients from a predetermined set (usually the set of integers; rational, real or complex numbers). Note that a constant is also a polynomial.

For example, these are polynomials:

$4x^2 + 6x - 9$, in the variable x,
$x^3 + 3x^2y + 3xy^2 + y^3$, in the variables x and y,
$5x^4 - 2x^2 + 9$, in the variable x,
$\sin^2 x + 5$, in the variable $\sin x$,
35, in any variable

A *root* is a value for a variable that will make the polynomial equal zero. For an example, 2 is a root of x^2-4 because $2^2-4 = 0$. For some polynomials, you can easily set the polynomial equal to zero and solve or otherwise find roots, but in some cases it is much more complicated.

Warm-up Problems

Problem 1. Find all roots of $x^2 + 2x + 1 = 0$.

Problem 2. Find all roots of $x^3 + 3x^2 - 4x - 12 = 0$.

Problem 3.
(a) If $p(x)$ is a quadratic polynomial with $p(1) = 1, p(2) = 2, p(3) = 3$, find $p(4)$.
(b) If $p(x)$ is a cubic polynomial with $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$ find $p(6)$.

More Difficult Problems

Problem 4. Show that

$$\sqrt[3]{20 + 14\sqrt{2}} + \sqrt[3]{20 - 14\sqrt{2}} = 4$$

Problem 5. If

$$x^5 + 5x^4 + 10x^3 + 10x^2 - 5x + 1 = 10,$$

and $x \neq -1$, compute the numerical value of $(x+1)^4$.

Problem 6. Compute the value of the expression

$$2024^4 - 4 \times 2022^4 + 6 \times 2020^4 - 4 \times 2018^4 + 2016^4$$

without using calculator.

Problem 7. Compute $\sqrt{28 \cdot 27 \cdot 26 \cdot 25 + 1}$ without using calculator.

Problem 8. Find all the roots of the polynomial

$$x^5 - 5x^4 + 11x^3 - 13x^2 + 9x - 3.$$

Problem 9. Find the real root of $x^5 + 5x^3 + 5x - 1$.

Problem 10. Let a and b be real numbers such that

$$a^3 - 15a^2 + 20a - 50 = 0 \quad \text{and} \quad 8b^3 - 60b^2 - 290b + 2575 = 0.$$

Compute $a + b$.

Problem 11. Find all real values of x such that

$$\sqrt{x + 2015} = x^2 - 2015.$$

Problem 12. Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$