**Mid-Cities Math Circle** $(MC)^2$
**Pigeonhole Principle**
**September 4, 2024**

## Warm-up Problems

**Problem 1.** Twenty five crates of apples are delivered to a store. The apples are of three different sorts, and all the apples in each crate are of the same sort. Show that among these crates there are at least nine containing the same sort of apple.

**Problem 2.** There are 52 people in a room. what is the largest value of $n$ such that the statement "At least $n$ people in this room have birthdays falling in the same month" is always true?

**Problem 3.** Can an equilateral triangle be covered completely by two smaller equilateral triangles? Justify your answer.

## More Difficult Problems

**Problem 4.** Ten students attempted to solve a total of 35 problems. Each problem was solved by one student only. There is at least one student who solved only one problem, at least one who solved only two problems and at least one who solved exactly three problems. Prove that there is also at least one student who has solved at least 5 problems.

**Problem 5.** Given any 5 distinct points on the surface of a sphere, show that we can find a closed hemisphere which contains at least 4 of them.

**Problem 6.** Show that in any group of 2024 people, there are two who have an identical number of friends within the group. (Assume friendship is a symmetric relationship.)

**Problem 7.** In a cube of side of length 9 there are 1981 points. Prove that there exist two of them situated at distance at most 1 from each other.

**Problem 8.** Show that among any six people, there are three who all know each other, or three who all do not know each other.

**Problem 9.** A sequence of $m$ positive integers contains exactly $n$ distinct terms. Show that if $2^n \leq m$, there exists a block of consecutive terms whose product is a perfect square.

**Problem 10.** Each of 9 given straight lines cuts a given square into two quadrilaterals whose areas are in proportion $2 : 3$. Prove that at least three of these lines pass through the same point.

**Problem 11.** In each of the unit squares of a $10 \times 10$ checkerboard, a positive integer not exceeding 10 is written. Any two numbers that appear in adjacent or diagonally adjacent squares of the board are relatively prime. Prove that some number appears at least 17 times.

**Problem 12.** Prove that among any ten points located inside a circle with diameter 5, there exist at least two at a distance less than 2 from each other.