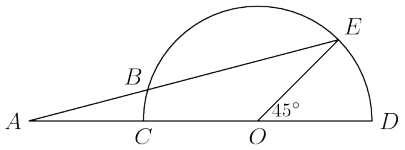
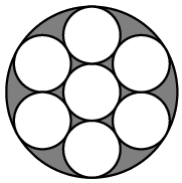
Mid-Cities Math Circle $(MC)^2$
Circle Geometry
April 17, 2024

Warm-up Problems

Problem 1. In the adjoining figure, CD is the diameter of a semi-circle with center O. Point A lies on the extension of DC past C; point E lies on the semi-circle, and B is the point of intersection (distinct from E) of line segment AE with the semi-circle. If length AB equals length OD, and the measure of $\measuredangle EOD$ is 45°, what is the measure of $\measuredangle BAO$?



Problem 2. Each of the small circles in the figure has radius one. The innermost circle is tangent to the six circles that surround it, and each of those circles is tangent to the large circle and to its small-circle neighbors. Find the area of the shaded region.



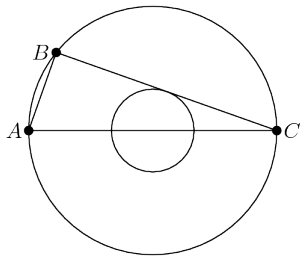
Problem 3. Two circles of equal radius r have an overlap area of 7π and the total area covered by the circles is 25π. What is the value of r?

More Difficult Problems

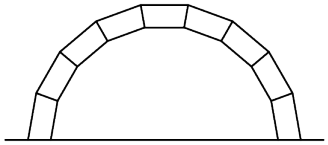
Problem 4. In triangle ABC, $AC = 24$ inches, $BC = 10$ inches, $AB = 26$ inches. Find the radius of the inscribed circle of triangle ABC.

Problem 5. One hundred concentric circles with radii $1, 2, 3, \ldots, 100$ are drawn in a plane. The interior of the circle of radius 1 is colored red, and each region bounded by consecutive circles is colored either red or green, with no two adjacent regions the same color. Find the ratio of the total area of the green regions to the area of the circle of radius 100.

Problem 6. The ratio of the radii of two concentric circles is $1 : 3$. If $\overline{AC}$ is a diameter of the larger circle, $\overline{BC}$ is a chord of the larger circle that is tangent to the smaller circle, and $AB = 12$, then what is the radius of the larger circle?



Problem 7. The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measure in degrees of the larger interior angle of the trapezoid. What is x?



Problem 8. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

Problem 9. Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).

Problem 10. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

Problem 11. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.