Mid-Cities Math Circle $(MC)^2$
Counting and Probability
April 4, 2024

Warm-up Problems

Problem 1. How many whole numbers from 1 through 46 are divisible by either 3 or 5 or both?

Problem 2. Each edge of a cube is colored either red or black. Every face of the cube has at least one black edge. What is the smallest possible number of black edges?

Problem 3. One thousand unit cubes are fastened together to form a large cube with edge length 10 units; this is painted and then separated into the original cubes. What is the number of these unit cubes which have at least one face painted?

More Difficult Problems

Problem 4. Henry's Hamburger Haven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties and any collection of condiments. How many different kinds of hamburgers can be ordered?

Problem 5. A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

Problem 6. A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability the grid is now entirely black?

Problem 7. Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

Problem 8. Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of one face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.

Problem 9. Let S be the set of positive integer divisors of 20^9. Three numbers are chosen independently and at random with replacement from the set S and labeled a_1, a_2, and a_3 in the order they are chosen. Find the probability that both a_1 divides a_2 and a_2 divides a_3.

Problem 10. Let $\mathcal{S}_n$ be the set of strings with only 0's or 1's with length n such that any 3 adjacent place numbers sum to at least 1. For example, 00100 works, but 10001 does not. Find the number of elements in $\mathcal{S}_{11}$.

Problem 11. A certain state issues license plates consisting of six digits (from 0 through 9). The state requires that any two plates differ in at least two places. (Thus the plates $\boxed{027592}$ and $\boxed{020592}$ cannot both be used.) Determine, with proof, the maximum number of distinct license plates that the state can use.

Problem 12. In a certain country there are n towns, where $n \geq 4$. A road may be built between towns A and B if there exist two other towns X and Y such that there is no road between towns A and X; there is no road between towns X and Y; there is no road between towns Y and B. What is the maximum number of roads that can be built?