Mid-Cities Math Circle $(MC)^2$
Combinatorics
May 1, 2024

Warm-up Problems

Problem 1: In how many ways can the letters in BEEKEEPER be rearranged so that two or more Es do not appear together?

Problem 2: For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

Problem 3: A single bench section at a school event can hold either 7 adults or 11 children. When N bench sections are connected end to end, an equal number of adults and children seated together will occupy all the bench space. What is the least possible positive integer value of N?

More Difficult Problems

Problem 4: On a given circle, six points A, B, C, D, E, and F are chosen at random, independently and uniformly with respect to arc length. Determine the probability that the two triangles ABC and DEF are disjoint, i.e., have no common points.

Problem 5: A positive integer divisor of 12! is chosen at random. The probability that the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Problem 6: Three distinct vertices are chosen at random from the vertices of a given regular polygon of $2n + 1$ sides. Assuming all such choices are equally likely, $P(n)$ is the probability that the center of the given polygon lies in the interior of the triangle determined by the three chosen random points. Compute $P(2024)$.

Problem 7: During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of mathematicians, there was some moment when both were asleep simultaneously. Prove that, at some moment, three of them were sleeping simultaneously. Assume that no two events can happen at the exact same time, and that all the mathematicians were awake at the start of the lecture.

Problem 8: A set of lines in the plane is in $general\ position$ if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its $finite\ regions$. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to color at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Problem 9: What is the largest number of towns that can meet the following criteria. Each pair is directly linked by just one of air, bus or train. At least one pair is linked by air, at least one pair by bus and at least one pair by train. No town has an air link, a bus link and a train link. No three towns, *A, B, C* are such that the links between *AB, AC,* and *BC,* are all air, all bus or all train.