Mid-Cities Math Circle $(MC)^2$
Recursion II
March 20, 2024

Warm-up Problems

Problem 1. A tree doubled its height every year until it reached a height of 32 feet at the end of 6 years. What was the height of the tree, in feet, at the end of 3 years?

Problem 2. Let p be a prime number. If p years ago, the ages of three children formed a geometric sequence with a sum of p and a common ratio of 2, compute the sum of the children's current ages.

Problem 3. Suppose the sequence $a_1, a_2, ...$ is defined as follows:

$$a_1 = 1, a_n = \frac{a_{n-1}}{a_{n-1}+1}.$$

Compute a_{2024}.

More Difficult Problems

Problem 4. For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots .$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

Problem 5. A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Find a_{2024}.

Problem 6. Let $a_1 = 1 + \sqrt{2}$ and for each $n \geq 1$ define $a_{n+1} = 2 - \frac{1}{a_n}$. Find the greatest integer less than or equal to the product $a_1 a_2 a_3 \cdots a_{200}$.

Problem 7. Let m be a positive integer, and let $a_0, a_1, \ldots, a_m$ be a sequence of real numbers such that $a_0 = 37, a_1 = 72, a_m = 0$, and

$$a_{k+1} = a_{k-1} - \frac{3}{a_k}$$

for $k = 1, 2, \ldots, m-1$. Find m.

Problem 8. Find $ax^5 + by^5$ if the real numbers $a, b, x,$ and y satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

Problem 9. Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?

Problem 10. Let $(x_n)_{n \geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

Problem 11. How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

Problem 12. Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.