Mid-Cities Math Circle $(MC)^2$
Recursion
February 28, 2024

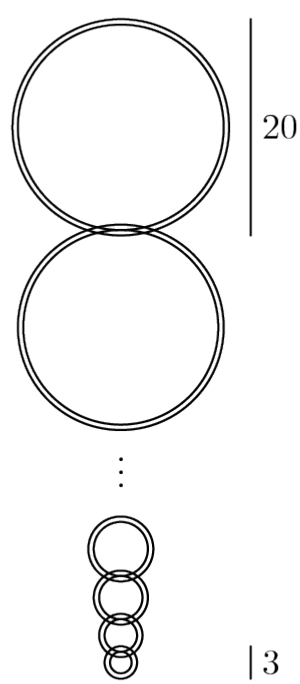
Warm-up Problems

Problem 1. Find the 5th term ($n = 5$) in the following recursive sequence:

$$a_1 = 3, a_n = 2a_{n-1} + 3.$$

Problem 2. A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

Problem 3. A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the outer rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?



More Difficult Problems

Problem 4. The degree measures of the angles in a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle.

Problem 5. Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and

$$t_n = \frac{5t_{n-1} + 1}{25t_{n-2}}$$

for all $n \geq 3$. Find t_{2024}.

Problem 6. The positive integers $x_1, x_2, ..., x_7$ satisfy $x_6 = 144$ and $x_{n+3} = x_{n+2}(x_{n+1} + x_n)$ for $n = 1, 2, 3, 4$. Find the last three digits of x_7.

Problem 7. Let $a_0 = 2$, $a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by 11. Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.

Problem 8. The sequence (a_n) satisfies $a_0 = 0$ and $a_{n+1} = \frac{8}{5}a_n + \frac{6}{5}\sqrt{4^n - a_n^2}$ for $n \geq 0$. Find the greatest integer less than or equal to a_{10}.

Problem 9. The sequence a_1, a_2, a_3, $\ldots$ of real numbers satisfies the recurrence

$$a_{n+1} = \frac{a_n^2 - a_{n-1} + 2a_n}{a_{n-1} + 1}.$$

Given that $a_1 = 1$ and $a_9 = 7$, find a_5.

Problem 10. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

Problem 11. How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

Problem 12. Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.