Mid-Cities Math Circle $(MC)^2$
Selected Number Theory Problems II
February 7, 2024

Warm-up Problems

Problem 1. The stronger Goldbach conjecture states that any even integer greater than 7 can be written as the sum of two different prime numbers. What is the largest possible difference between the two primes for such representations of the even number 126?

Problem 2. The increasing sequence $3, 15, 24, 48, \ldots$ consists of those positive multiples of 3 that are one less than a perfect square. What is the remainder when the 1994th term of the sequence is divided by 1000?

Problem 3. Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}}$ is divided by 1000.

More Difficult Problems

Problem 4. Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integral divisors, including 1 and itself. Find $\frac{n}{75}$.

Problem 5. For how many values of k is 12^{12} the least common multiple of the positive integers 6^6, 8^8, and k?

Problem 6. Let N be the greatest integer multiple of 8, no two of whose digits are the same. What is the remainder when N is divided by 1000?

Problem 7. Call a positive integer n extra-distinct if the remainders when n is divided by $2, 3, 4, 5,$ and 6 are distinct. Find the number of extra-distinct positive integers less than 1000.

Problem 8. Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$. Which of the following could be the value of $S(n+1)$?

(A) 1 **(B)** 3 **(C)** 12 **(D)** 1239 **(E)** 1265

Problem 9. The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find N.

Problem 10. When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)

Problem 11. Suppose that the set $\{1, 2, \cdots, 2024\}$ has been partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \leq i \leq 1012)$ so that for all i, $|a_i - b_i|$ equals 1 or 6. What is the last digit of the sum

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{1012} - b_{1012}|?$$

Problem 12. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.