Mid-Cities Math Circle $(MC)^2$
Selected Number Theory Problems
November 29, 2023

Warm-up Problems

Problem 1. Chris' birthday is on a Thursday this year. What day of the week will it be 60 days after her birthday?

Problem 2. Among the integers $1, 2, \ldots, 2023$, what is the maximum number of integers that can be selected such that the sum of any two selected numbers is not a multiple of 7?

Problem 3. Define a sequence recursively by $F_0 = 0,\ F_1 = 1$, and $F_n =$ the remainder when $F_{n-1} + F_{n-2}$ is divided by 3, for all $n \geq 2$. Thus the sequence starts $0, 1, 1, 2, 0, 2, \ldots$ What is

$$F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}?$$

More Difficult Problems

Problem 4. Find the least positive integer n such that no matter how 10^n is expressed as the product of any two positive integers, at least one of these two integers contains the digit 0.

Problem 5. Find the sum of all positive two-digit integers that are divisible by each of their digits.

Problem 6. What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

Problem 7. In Pascal's Triangle, each entry is the sum of the two entries above it. The first few rows of the triangle are shown below.

Row 0:							1							
Row 1:						1		1						
Row 2:					1		2		1					
Row 3:				1		3		3		1				
Row 4:			1		4		6		4		1			
Row 5:		1		5		10		10		5		1		
Row 6:	1		6		15		20		15		6		1	

In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3:4:5$?

Problem 8. What is the greatest integer less than or equal to

$$\frac{3^{100}+2^{100}}{3^{96}+2^{96}}?$$

Problem 9. Given that 2^{2004} is a 604-digit number whose first digit is 1, how many elements of the set $S=\{2^0,2^1,2^2,\ldots,2^{2003}\}$ have a first digit of 4?

Problem 10. How many nonnegative integers can be written in the form

$$a_7\cdot 3^7+a_6\cdot 3^6+a_5\cdot 3^5+a_4\cdot 3^4+a_3\cdot 3^3+a_2\cdot 3^2+a_1\cdot 3^1+a_0\cdot 3^0,$$

where $a_i\in\{-1,0,1\}$ for $0\le i\le 7$?

Problem 11. Suppose that the set $\{1,2,\cdots,2024\}$ has been partitioned into disjoint pairs $\{a_i,b_i\}$ $(1\le i\le 1012)$ so that for all i, $|a_i-b_i|$ equals 1 or 6. What is the last digit of the sum

$$|a_1-b_1|+|a_2-b_2|+\cdots+|a_{1012}-b_{1012}|?$$

Problem 12. Find all solutions to $(m^2+n)(m+n^2)=(m-n)^3$, where m and n are non-zero integers.