Mid-Cities Math Circle $(MC)^2$
Sums, Products, Sequences
November 1, 2023

Warm-up Problems

Problem 1. Evaluate $1+2-3-4+5+6-7-8+\cdots+2021+2022-2023$.

Problem 2. The expressions $A = 1\times 2+3\times 4+5\times 6+\cdots+37\times 38+39$ and $B = 1+2\times 3+4\times 5+\cdots+36\times 37+38\times 39$ are obtained by writing multiplication and addition operators in an alternating pattern between successive integers. Find the positive difference between integers A and B.

Problem 3. Find the product

$$\frac{8}{4}\cdot\frac{12}{8}\cdot\frac{16}{12}\cdot\dots\cdot\frac{4n+4}{4n}\cdot\dots\cdot\frac{2028}{2024}$$

More Difficult Problems

Problem 4. Find the sum $\frac{1}{1(3)}+\frac{1}{3(5)}+\frac{1}{5(7)}+\cdots+\frac{1}{255(257)}$.

Problem 5. Determine the value of the sum

$$\frac{3}{1^2\cdot 2^2}+\frac{5}{2^2\cdot 3^2}+\frac{7}{3^2\cdot 4^2}+\cdots+\frac{29}{14^2\cdot 15^2}.$$

Problem 6. A palindrome, such as 83438, is a number that remains the same when its digits are reversed. The numbers x and $x+32$ are three-digit and four-digit palindromes, respectively. What is the sum of the digits of x?

Problem 7. The sequence $\{a_n\}$ is defined by

$$a_0 = 1, a_1 = 1, \text{ and } a_n = a_{n-1}+\frac{a_{n-1}^2}{a_{n-2}} \text{ for } n\geq 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0 = 1, b_1 = 3, \text{ and } b_n = b_{n-1}+\frac{b_{n-1}^2}{b_{n-2}} \text{ for } n\geq 2.$$

Find $\frac{b_{32}}{a_{32}}$.

Problem 8. Find all sets of two or more consecutive positive integers whose sum is 100.

Problem 9. For $n \geq 3$, let $f(n)$ be the number of subsets of three elements that can be chosen from a set of n distinct elements. Compute

$$\sum_{n=3}^{101} \frac{1}{f(n)}.$$

Problem 10. Express $\sum_{k=1}^{\infty} 6^k/((3^{k+1}-2^{k+1})(3^k-2^k))$ as a rational number.

Problem 11. Let F_n be the nth Fibonacci number, where as usual $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Find the value of the infinite sum

$$\frac{1}{3} + \frac{1}{9} + \frac{2}{27} + \cdots + \frac{F_n}{3^n} + \cdots .$$

Problem 12. Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

Mid-Cities Math Circle $(MC)^2$
Sums, Products, Sequences II
November 15, 2023

Warm-up Problems

Problem 1. Let N be the smallest positive integer whose sum of its digits is 2023. What is the sum of the digits of $N + 2023$?

Problem 2. Prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \frac{1}{2023} - \frac{1}{2024} = \frac{1}{1013} + \frac{1}{1013} + \cdots + \frac{1}{2024}.$$

Problem 3. Suppose that (u_n) is a sequence of real numbers satifying $u_{n+2} = 2u_{n+1} + u_n$, and that $u_3 = 9$ and $u_6 = 128$. What is u_5?

More Difficult Problems

Problem 4. The first two terms of a sequence are $a_1 = 1$ and $a_2 = \frac{1}{\sqrt{3}}$. For $n \geq 1$,

$$a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}}.$$

What is a_{2009}?

Problem 5. Find

$$\left(1 + \frac{1}{1+2^1}\right)\left(1 + \frac{1}{1+2^2}\right)\left(1 + \frac{1}{1+2^3}\right)\cdots\left(1 + \frac{1}{1+2^{10}}\right).$$

Problem 6. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Find

$$\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}.$$

Problem 7. Let $a_0, a_1, \ldots$ be a sequence such that $a_0 = 3$, $a_1 = 2$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$. Find

$$\sum_{n=0}^{8} \frac{a_n}{a_{n+1}a_{n+2}}.$$

Problem 8. Let A be the set of positive integers that have no prime factors other than 2, 3, or 5. Evaluate the infinite sum

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{12}+\frac{1}{15}+\frac{1}{16}+\frac{1}{18}+\frac{1}{20}+\cdots$$

of the reciprocals of all the elements of A.

Problem 9. Let m be a positive integer, and let $a_0, a_1, \ldots, a_m$ be a sequence of real numbers such that $a_0 = 37, a_1 = 72, a_m = 0$, and

$$a_{k+1} = a_{k-1} - \frac{3}{a_k}$$

for $k = 1, 2, \ldots, m-1$. Find m.

Problem 10. A sequence of real numbers x_n is defined recursively as follows: x_0, x_1 are arbitrary positive real numbers, and

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}, \quad n = 0, 1, 2, \ldots$$

Find x_{2023} in terms of x_0 and x_1.

Problem 11. Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2},$$

for $i = 1, 2, \ldots, n$.

Old Problems

Problem 12. For $n \geq 3$, let $f(n)$ be the number of subsets of three elements that can be chosen from a set of n distinct elements. Compute

$$\sum_{n=3}^{101} \frac{1}{f(n)}.$$

Problem 13. Express $\sum_{k=1}^{\infty} 6^k/((3^{k+1}-2^{k+1})(3^k-2^k))$ as a rational number.

Problem 14. Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$