Mid-Cities Math Circle $(MC)^2$
Selected Problems with Focus on Lengths, Areas and Volumes
October 18, 2023

Old Problems

Problem 7. Let $\overline{AB}$ be a diameter of circle ω. Extend $\overline{AB}$ through A to C. Point T lies on ω so that line CT is tangent to ω. Point P is the foot of the perpendicular from A to line CT. Suppose $\overline{AB} = 18$. Find the maximum possible length of segment BP.

Problem 8. Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

Problem 9. Tetrahedron $ABCD$ has $AD = BC = 28$, $AC = BD = 44$, and $AB = CD = 52$. For any point X in space, suppose $f(X) = AX + BX + CX + DX$. Find the least possible value of $f(X)$.

New Problems

Problem 10. For how many values of n will an n-sided regular polygon have interior angles with integral measures?

Problem 11. Triangle ABC is isosceles with $AC = BC$ and $\angle ACB = 106°$. Point M is in the interior of the triangle so that $\angle MAC = 7°$ and $\angle MCA = 23°$. Find $\angle CMB$.

Problem 12. Square $ABCD$ has sides of length 2. Set S is the set of all line segments that have length 2 and whose endpoints are on adjacent sides of the square. The midpoints of the line segments in set S enclose a region whose area is k. Find k.

Problem 13. Triangle ABC has $\angle C = 60°$ and $BC = 4$. Point D is the midpoint of BC. What is the largest possible value of $\tan \angle BAD$?

Problem 14. Let S be a square with the side length 20 and let M be the set of points formed with the vertices of S and another 1999 points lying inside S. Prove that there exists a triangle with vertices in M and with area at most equal with $\frac{1}{10}$.

Problem 15. Two distinct regular tetrahedra have all their vertices among the vertices of the same unit cube. What is the volume of the region formed by the intersection of the tetrahedra?

Problem 16. To clip a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon P_6 of area 1 is clipped to obtain a heptagon P_7. Then P_7 is clipped (in one of the seven possible ways) to obtain an octagon P_8, and so on. Prove that no matter how the clippings are done, the area of P_n is greater than $\frac{1}{3}$, for all $n \geq 6$.