UT Arlington Mid-Cities Math Circle $(MC)^2$
Pigeonhole Principle
September 20, 2023

Classical pigeonhole principle: If n objects are arrenged in m pigeonholes with $n > m$, then at least one pigeonhole must contain more than one object (objects=pigeons).

Strong pigeonhole principle: If n objects are arranged in k pigeonholes, then there is a pigeonhole that contains at least $\lceil \frac{n}{k} \rceil$ objects. ($\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x.)

Infinite pigeonhole principle: Given an infinite set of objects, if they are arranged in a finite number of pigeonoholes, there is at least one pigeonhole with an infinite number of objects.

Warm-up problems.

Problem 1. Peter has 7 pairs of socks in his drawer, one of each color of the rainbow. How many socks does he have to draw out in order to guarantee that he has grabbed at least one pair? What if there are likewise colored pairs of gloves in there and he cannot tell the difference between gloves and socks and he wants a matching set?

Problem 2. Given 50 distinct positive integers strictly less than 99, prove that there are two numbers whose sum is 99.

Problem 3. Given nine points inside the unit square, prove that some three of them form a triangle whose area does not exceed 1/8.

More difficult problems.

Problem 4. Over a 44 day period, Mary will train for triathlons at least once per day, and a total of 70 times in all. Show that there is a period of consecutive days during which she trains exactly 17 times.

Problem 5. There are 25 students in the math circle session and the sum of their ages is 514 years. Show that there are 17 students such that the sum of their ages is at least 350 years.

Problem 6. There are 2023 people who shake hands with one another. Prove that there is a pair of people who will shake hands with the same number of people.

Problem 7. Draw the diagonals of a 21-gon. Prove that at least one angle of less than 1° is formed.

Problem 8. Let A be any set of 20 distinct integers chosen from the arithmetic progression $\{1, 4, 7, ..., 100\}$. Prove that there must be two distinct integers in A whose sum if 104.

Problem 9. There are 51 senators in a senate. The senate needs to be divided into n committees such that each senator is on exactly one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does "not" necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

Problem 10. Five points lie in an equilateral triangle of side length 1. Show that two of the points lie no further than 1/2 apart. Can "1/2" be replaced by a smaller number? Can we improve the problem by replacing five points with four points?

Problem 11. The sides of a regular triangle $\triangle ABC$ are colored in two colors. Do there exist three monochromatic points on the perimeter of $\triangle ABC$ that form a right triangle?

Problem 12. A class of 32 students is organized in 33 teams. Every team consists of three students and there are no identical teams. Show that there are two teams with exactly one common student.