Mid-Cities Math Circle $(MC)^2$
Games
September 13, 2023

Warm-up Problems

Problem 1. Initially there are 20 checkers on the table. Two players take turns removing 1, 2, 3, or 4 checkers. The winner is the one who removes the last checker. Who has a winning strategy?

Problem 2. Initially there are 2023 checkers on the table. Two players take turns removing 1, 2, 3, 4, or 5 checkers. The winner is the one who removes the last checker. Who has a winning strategy?

Problem 3. On a table there are 20 coins. Two players take turns removing coins from the table. In each turn they can remove $2, 5$ or 6 coins. The first one that cannot make a move loses. Who has a winning strategy?

More Difficult Problems

Problem 4. There is a row of 10 coins on the table; each coin can have some positive integer value. Two players alternate turns. On each turn a player must take one of the two coins on either end of the row of remaining coins, so with each turn the row gets shorter by one. After all the coins have been taken, the player with the higher total value is the winner. Who can win the game? How about 100 coins?

Problem 5. On a 2023×2024 rectangular chessboard there is a stone in the lower leftmost square. Players A and B move the stone alternately, starting with A. In each step one can move the stone upward or rightward any number of squares. The player who moves it into the upper rightmost square wins. Who has a winning strategy?

Problem 6. Two players A and B play the following game. First A says 1, 2 or 3. Then B can add 1, 2 or 3 to the number the first player said. The game continues with the players playing alternately, in each turn adding 1, 2 or 3 to the previous number. For example, A can say 2, then B can say 5, then A could say 6, and so on. The player who says 2023 wins. Who has the winning strategy?

Problem 7. Consider an 8×8 board. Initially all the squares are painted white and black like in a chessboard. The allowed operation is to choose two unit squares that share one side and recolor them in the following way: Any white square is painted black, any black square is painted green and any green square is painted white. Determine, if it is possible, using this operation several times, to get all the original black squares to be painted white and all the original white squares to be painted black. What about a 9×9 board?

Note: Initially there are no green squares, but these appear after the first time we use the operation.

Problem 8. Two players play a game on an infinite board that consists of 1×1 squares. Player I chooses a square and marks it with an O. Then, Player II chooses another square and marks with an X. They play until one of the players marks a row or a column of five consecutive squares, and this player wins the game. If no player can achieve this, the game is a tie. Show that Player II can prevent Player I from winning.

Problem 9. Fred Flintstone and Barney Rubble in turn add pebbles to a pile. On each turn, they must add at least one pebble and may not add more pebbles than there are already in the pile. The player who makes the pile consist of exactly 2023 pebbles wins. Find a strategy that allows Fred or Barney to win regardless of how the other may play. Oh, there is originally just one pebble in the pile, and Fred goes first.

Problem 10. Players A and B take turns writing a number as follows. First A writes the number 1, and then B writes 2. Hereafter, in each move, if the current number is k, then the player whose turn it is can either write $k + 1$ or $2k$, but no player can write a number larger than 2023. The player who writes 2023 wins. Who has a winning strategy? What if 2023 is replaced by 2024?

Problem 11. The SOS 2024 Game is played on a 1×2024 grid as follows. Two players in turn write either an S or an O in an empty square.The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing any SOS, then the game is a draw. Prove that the second player has a winning strategy.