UT Arlington Mid-Cities Math Circle $(MC)^2$
Problems Involving Digits
November 3, 17, 2021

Warm-up problems

Problem 1. Let $N = 9 + 99 + 999 + \cdots + 999...9$, where the last term in that sum has 321 nines. Determine the sum of the digits of N.

Problem 2.

(a) Does there exist a positive integer n such that the sum of the digits of n^2 is equal to 2021?

(b) What if we change 2021 to 2020?

More difficult problems

Problem 3. Find the smallest positive integer n, so that $999999 \cdot n = 111...11$.

Problem 4. Find the smallest positive integer ending in 1986 which is divisible by 1987.

Problem 5. Show that 1982 divides 222...22 (1980 twos).

Problem 6. The integers $1, ..., 1986$ are written in any order and concatenated. Show that the result is always an integer which is not the cube of another integer.

Problem 7. Prove that the digit which is the prior to the last digit in the decimal expansion of 3^n is even.

Problem 8. Let a be a nonnegative integer. Let b be the product of the digits of a, and let c be the product of the digits of b. Suppose that $b \geq 10$, and c is odd. Determine all possible values of the units digit of b.

Problem 9. If n is a positive integer such that the first digit in the decimal expansion of both 2^n and 5^n is x, find x.

Problem 10. Find the first digit before and after the decimal point of $(\sqrt{2}+\sqrt{3})^{1980}$.

Problem 11. Does there exist a nonconstant polynomial $P(x)$ with integer coefficients, such that, for every positive integer n, the sum of the digits of $|P(n)|$ is not a Fibonacci number? (Recall that the Fibonacci numbers are the elements of the sequence $0, 1, 1, 2, 3, 5, ...$, where each term after the first two terms is the sum of the two terms before it.)

Problem 12. Let n be a positive integer number such that $4^n + 2^n + 1$ is a perfect square. Prove that n is a power of 3.

Problem 13. Prove that there are infinitely many composite numbers n, such that n divides $3^{n-1} - 2^{n-1}$.

Problem 14. Prove that there exist infinitely many positive integers n such that the decimal expansion of 2^n ends with n.