Introduction to Complex Numbers, at the UTA $(MC)^2$

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1 Warm-up Problems

Problem 1. Find the value of $\frac{1+7i}{4+3i}$.

Problem 2. Find the value of $(1+i)^{12}$.

2 Problems

Problem 3. Define a sequence $z_0, z_1, z_2, \ldots$ of complex numbers recursively by $z_0 = \frac{1}{100} + i$ and $z_{n+1} = \frac{z_n + i}{z_n - i}$ for all nonnegative integers n. Let a and b be real numbers such that $z_{1000} = a + bi$. Find $a + b$.

Problem 4. Find the number of complex numbers z such that $|z| = 1$ and $z^{6!} - z^{5!}$ is real.

Problem 5. Find the number of ordered pairs (x, y) of integers with $1 \le x < y \le 1000$ such that $i^x + i^y$ is real.

Problem 6. Find the number of ordered quadruples (a, b, c, d) of complex numbers such that, for all complex numbers x and y, we have that $(ax + by)^3 + (cx + dy)^3 = x^3 + y^3$.

Problem 7. Find all complex numbers z such that $12|z|^2 = 2|z+2|^2 + |z^2+1|^2 + 31$.

Problem 8. Let P be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have positive imaginary part. Suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \le \theta < 360$. Find θ.

3 More Difficult Problems

Problem 9. Find all nonconstant polynomials $P(z)$ with complex coefficients for which all complex roots of the polynomials $P(z)$ and $P(z) - 1$ have absolute value 1.

Problem 10. Let w, x, y, and z be complex numbers with $|w| = |x| = |y| = |z| = 1$ and $wxyz + 3 = w + x + y + z$. Prove that at least one of w, x, y, and z is equal to 1.