Areas and Geometry, at the UTA $(MC)^2$

Luke Robitaille

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Warm-up Problems

Problem 1. Find the area of a triangle with sides $3, 4$, and 5.

Problem 2. Find the area of a regular hexagon with side $\sqrt{3}$.

Problem 3. Find the area of a kite with diagonals x and y.

More Difficult Problems

Problem 4. Suppose that $DIAL$, FOR, and $FRIEND$ are regular polygons in the plane. Given that $ID = 1$, find the product of all possible values of the area of triangle OLA.

Problem 5. Equilateral triangle ABC has area 1. Suppose that A',B', and C' are the midpoints of sides BC, AC, and AB, respectively. Suppose that A'', B'', and C'' are the midpoints of segments $B'C'$, $A'C'$, and $A'B'$, respectively. Find the area of quadrilateral $BB''C''C$.

Problem 6. In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.

Problem 7. A right prism with height h has bases that are regular hexagons with sides of length 12. A vertex A of the prism and its three adjacent vertices are the vertices of a triangular pyramid. The dihedral angle (the angle between the two planes) formed by the face of the pyramid that lies in a base of the prism and the face of the pyramid that does not contain A has measure $60°$. Find h^2.

Problem 8. A kite is inscribed in a circle with center O and radius 60. The diagonals of the kite meet at a point P, and OP is an integer. Find the smallest possible value of the area of the kite.

Problem 9. In $\triangle ABC, AB = AC = 10$ and $BC = 12$. Let x be a real number. Suppose that point D lies strictly between A and B on $\overline{AB}$ and point E lies strictly between A and C on $\overline{AC}$ so that $AD = DE = EC = x$. What is the value of x?

Problem 10. Let ABC be a triangle with $AB = 5$, $AC = 8$, and $\angle BAC = 60°$. Let $UVWXYZ$ be a regular hexagon that is inscribed inside triangle ABC such that U and V lie on side AB, W and X lie on side AC, and Z lies on side BC. What is the side length of hexagon $UVWXYZ$?

Problem 11. Point B lies on line segment $\overline{AC}$ with $AB = 16$ and $BC = 4$. Points D and E lie on the same side of line AC forming equilateral triangles $\triangle ABD$ and $\triangle BCE$. Let M be the midpoint of $\overline{AE}$, and N be the midpoint of $\overline{CD}$. What is the area of $\triangle BMN$?

Problem 12. Two real numbers a and b are chosen independently and uniformly at random from the interval $(0, 75)$. Let O and P be two points in the plane with $OP = 200$. Let Q and R be on the same side of line OP such that the degree measures of $\angle POQ$ and $\angle POR$ are a and b respectively, and $\angle OQP$ and $\angle ORP$ are both right angles. What is the probability that $QR \leq 100$?

Problem 13. Circle C with radius 2 has diameter $\overline{AB}$. Circle D is internally tangent to circle C at A. Circle E is internally tangent to circle C, externally tangent to circle D, and tangent to $\overline{AB}$. The radius of circle D is three times the radius of circle E. Find the radius of circle D.

Problem 14. Let $ABCDEF$ be a regular hexagon with side length 2. A circle γ with radius 3 and center at A is drawn. Find the area of the region that lies inside quadrilateral $BCDE$ and outside γ.

Problem 15. Given a circle of radius $\sqrt{13}$, let A be a point at a distance $4+\sqrt{13}$ from the center O of the circle. Let B be the point on the circle nearest to point A. A line passing through the point A intersects the circle at points K and L. What is the maximum possible value of the area of $\triangle BKL$?