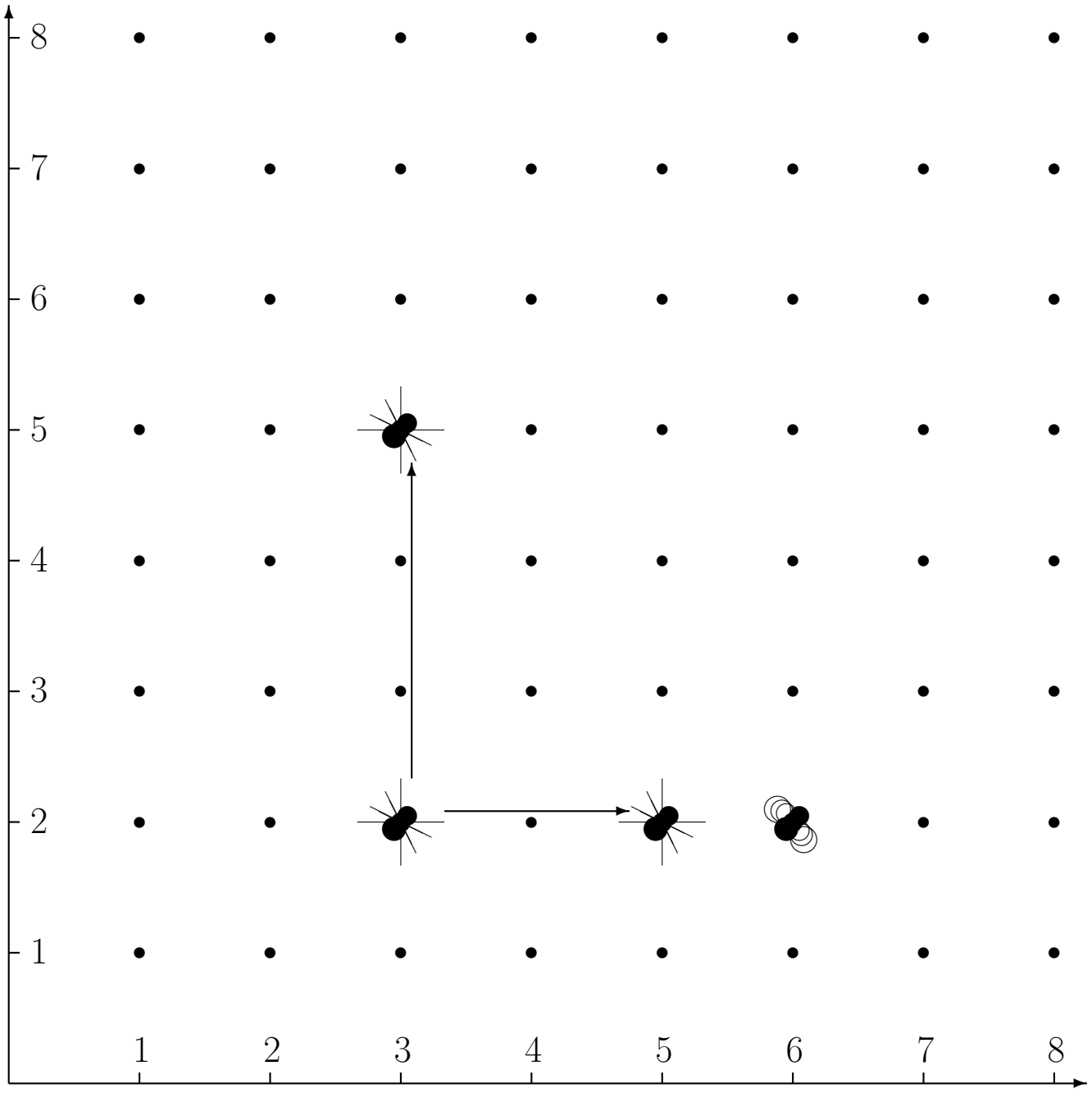
Mid-Cities Math Circle 1/23/14

The Spider and the Fly

A spider sits at the point with integer coordinates (a, b) in the positive quadrant of the coordinate plane. This is no ordinary spider. If the spider sits at (a, b) it can jump to either $(a + b, b)$ or to $(a, a + b)$. The spider pictured below sits at the point $(3, 2)$. It can therefore move to $(5, 2)$ or $(3, 5)$.

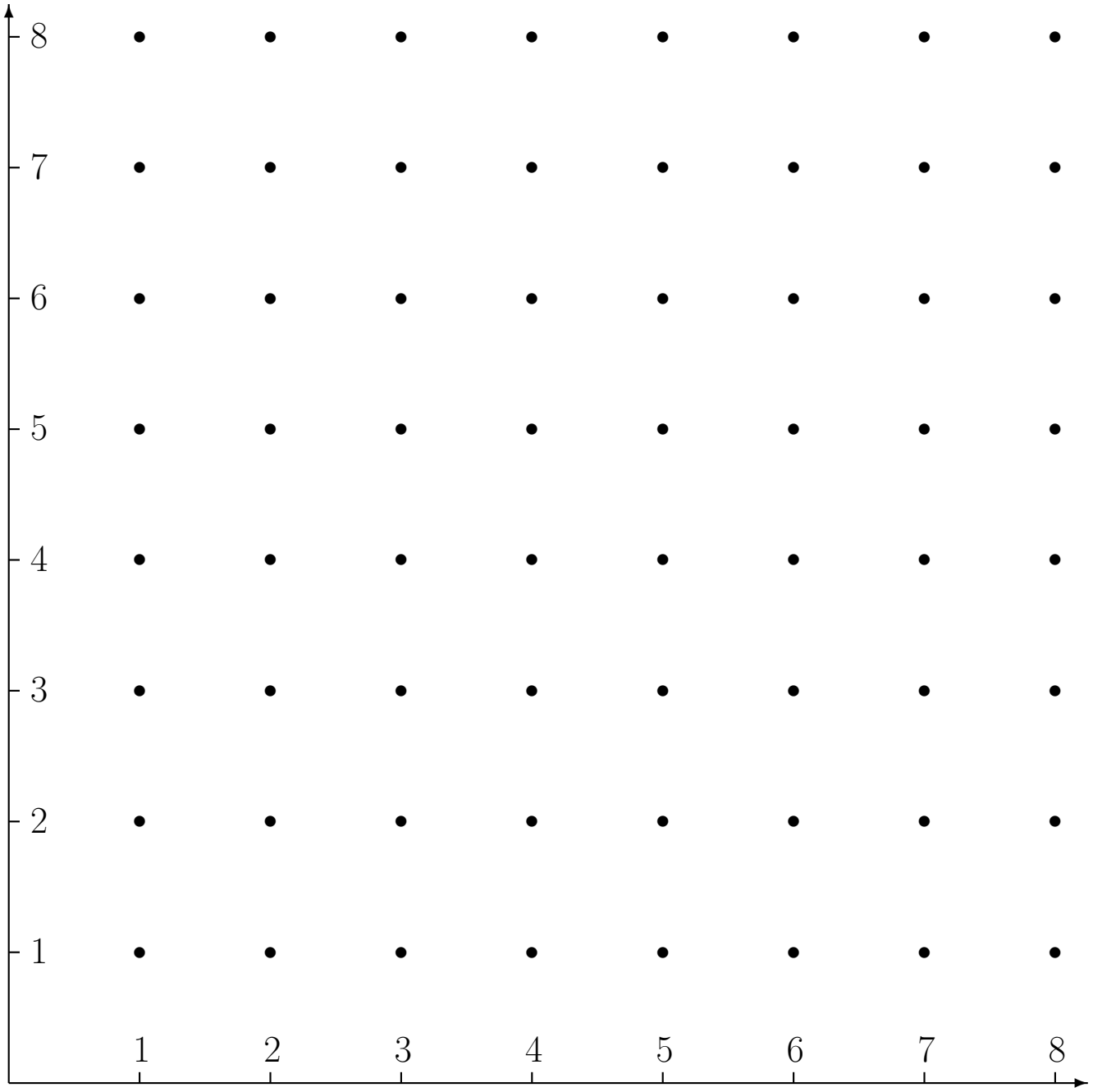


Questions

1. The spider starts at $(1, 1)$. Can it reach the fly, sitting at $(6, 2)$?

2. The spider starts at $(1, 1)$. Can you find safe places for the fly, points with integer coordinates that the spider cannot reach?

3. On the picture below circle all the points that the spider starting at $(1, 1)$ can reach. Can you find a pattern for the coordinates of the points that the spider can reach?



4. The spider starts at $(1, 1)$. Can it reach the fly, sitting at $(3169, 6903)$?

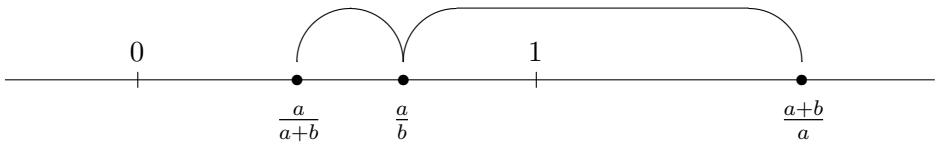
5. Determine all the squares that the spider can reach if it starts at $(2, 2)$. What about spiders starting at $(3, 3)$, $(4, 4)$ etc. ? How many of Euclid's spiders do we need to cover all the points with integer coordinates in the positive quadrant?

6. Assume that the spider started at $(1, 1)$ and reached the square (a, b). How many different paths are there from $(1, 1)$ to (a, b)?

7. This is a new breed of advanced spider. It can move four different ways. Starting from a point (a, b), the spider can move to $(a + b, b)$, $(a - b, b)$, $(a, a + b)$, or $(a, b - a)$. The spider sits at $(-2561, 2353)$ and the fly sits at $(3169, 6903)$. The fly is terrified and sits perfectly still. Will the spider ever catch the fly?

8. The fly has been caught by the spider. The spider likes to play with its prey. The spider says:" Dear Fly, my favorite number is 51. I will write it on the board. We will take turns subtracting a proper divisor from the number that is written on the board and replacing it by the resulting smaller number. The winner is the last player able to perform such a subtraction. The loser is the player left with the a number that has no proper divisor, the number 1. If you win, I will spare your life." Can the fly be saved?

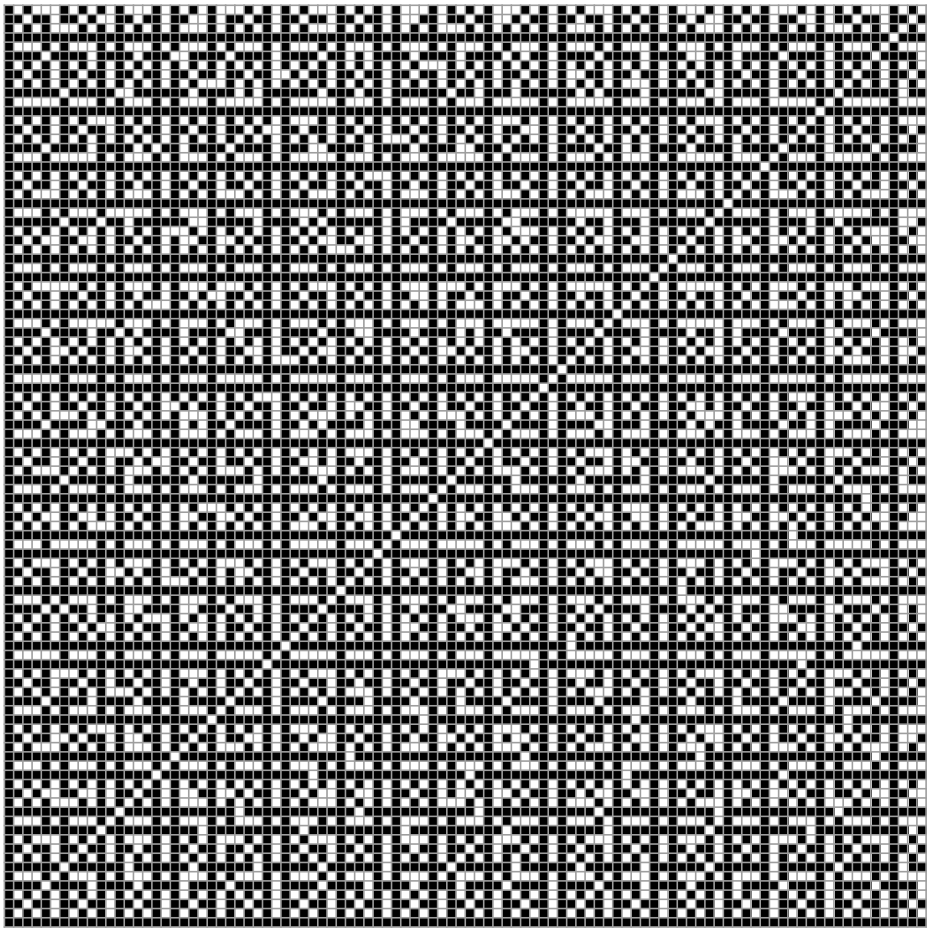
9. **The Rational Spider** If the rational spider sits at the point $\frac{a}{b}$ on the positive part of the number line it can jump either to the point $\frac{a+b}{b}$ or to the point $\frac{a}{a+b}$. Are there any points on the positive part of the number line for the fly to hide, if the rational spider sits at 1?

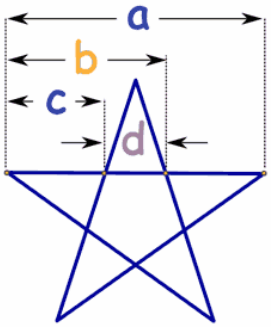


10. **The Zig-Zag Spider** The rational spider starts at 1, first jumps to the left, then to the right, to the left again, to the right and just keeps on going. Where will the spider end up?

11. Once again the spider has caught the fly. It offers the fly another game. The spider writes the integers 1, 2, 3, ... , 2012, 2013, 2014 on the board. "We will alternately erase one integer at a time until only two integers are left on the board. If the two remaining integers have no common factor other than 1, you win the game and I will spare your life. Otherwise I will have you for lunch. I am the spider and therefore I will go first." Find a winning strategy for the fly.

12. The spider offers another game to the fly. "I will choose a number, maybe 36, and list all its positive divisors including 1 and 36. We will take turns crossing out a number from that list. Each time we cross out a number, we must also cross out all of its divisors. Whoever is forced to cross out 36 is the loser of the game. I will let you go first. If you lose I will have you for dinner."





Parthenon in Athens, Greece