UT Arlington Mid-Cities Math Circle $(MC)^2$
Fibonacci Numbers and Beyond

The "classical" Fibonacci numbers are defined by the Fibonacci sequence:

$$a_1 = a_2 = 1, a_{n+2} = a_{n+1} + a_n, n \geq 1.$$

Problem 1. Prove that (a)

$$a_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}.$$

(b) $a_1 + a_2 + ... + a_n = a_{n+2} - 1$.

Problem 2. Prove that, for any n, there is a Fibonacci number ending with n zeros.

Problem 3. How many subsets of $\{1, 2, ..., n\}$ have no two successive numbers?

Problem 4. Consider a row of n seats. There are n children and each child sits on one seat. Then each child may move to a neighbor seat. Find the number of all possible children rearrangements.

Problem 5. Same as in Problem 3, but consider a circle of n chairs.

Problem 6. Prove that

$$\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \cdots + \frac{a_n}{2^n} < 2$$

for every $n > 0$.

Problem 7. In how many ways can you tile a $2 \times n$ rectangle by 2×1?

Problem 8. In how many ways can you tile a $2 \times n$ rectangle by 2×1 or 2×2 tiles?

Problem 9. In how many ways can you tile a $3 \times n$ rectangle by 2×1?