UT Arlington Mid-Cities Math Circle $(MC)^2$
Number Theory Problems

"Mathematics is the queen of the sciences and number theory is the queen of mathematics" –Gauss

Problem 1. Find the largest prime factor of:

$$3(3(3(3(3(3(3(3(3(3+1)+1)+1)+1)+1)+1)+1)+1)+1)+1$$

Problem 2. The Fibonacci sequence is defined as follows: $F_0 = 0, F_1 = 1$, and if $n > 1$, $F_n = F_{n-1} + F_{n-2}$. The first few terms in the sequence are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...$$

For which n is F_n divisible by 2? by 3? by 5? by 7? by 11?

Problem 3. Prove that two consecutive Fibonacci numbers are always relatively prime.

Problem 4. Find all natural numbers x, y, and z satisfying the following equation

$$x^x + y^y = z^z$$

(Reminder: A number n is natural if it is a positive integer.)

Problem 5. (a) Prove that there are infinitely many prime numbers.
(b) Prove that there are infinitely many prime numbers of the form $4n + 3$.

Problem 6. Suppose $n > 1$ is an integer. Show that $n^4 + 4^n$ is not prime.

Problem 7. Prove that there are no primes in the following infinite sequence of numbers:

$$1001, 1001001, 1001001001, 1001001001001, ...$$

Problem 8. Show that there exist 2011 consecutive numbers, each of which is divisible by the cube of an integer.