UT Arlington Mid-Cities Math Circle $(MC)^2$
Pigeonhole Principle

The pigeonhole principle states that if n items are put into m pigeonholes with $n > m$, then at least one pigeonhole must contain more than one item (items=pigeons).

Warm-up 1. If each point of the plane is colored red or blue then there are two points of the same color at distance 1 from each other.

Warm-up 2. Among 13 persons, there are two born in the same month.

Problem 1. If there are n number of people who can shake hands with one another (where $n > 1$), then there is always a pair of people who will shake hands with the same number of people.

Problem 2. Prove that however one selects 55 integers $1 \leq x_1 < x_2 < ... < x_{55} \leq 100$, there will be some two that differ by 9, some two that differ by 10, a pair that differ by 12, and a pair that differ by 13. Surprisingly, there need not be a pair of numbers that differ by 11.

Problem 3. Prove that any $(n+1)$-element subset of $\{1, 2, ..., 2n\}$ contains two integers that are relatively prime.

Problem 4. (Putnam Exam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression $\{1, 4, 7, ..., 100\}$. Prove that there must be two distinct integers in A whose sum if 104.

Problem 5. During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Problem 6. Prove that among any seven real numbers $y_1, ..., y_7$, there are two, y_i and y_j, such that

$$0 \leq \frac{y_i - y_j}{1 + y_i y_j} \leq \frac{1}{\sqrt{3}}.$$

Problem 7. Prove that among five different integers there are always three with sum divisible by 3.