UT Arlington Mid-Cities Math Circle $(MC)^2$
Algebra Problems II

Problem 8. (AMC10, 2002) The number $25^{64} \cdot 64^{25}$ is the square of a positive integer N. What is the sum of the digits of N in its decimal representation?

Problem 9. (AIME, 2002) Consider the sequence defined by $a_k = \frac{1}{k^2+k}$ for $k \geq 1$. Given that $a_m + a_{m+1} + \cdots + a_{n-1} = \frac{1}{29}$, for positive integers m and n with $m < n$, find $m + n$.

Problem 10. (AIME, 2004) Given that $\log_{10} \sin x + \log_{10} \cos x = -1$ and that $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, find n.

Problem 11. (AIME, 2004) How many positive integers less than $10,000$ have at most two different digits?

Problem 12. (AIME, 2003) A sequence of positive integers with $a_1 = 1$ and $a_9 + a_{10} = 646$ is formed so that the first three terms are in geometric progression, the second, third, and fourth terms are in arithmetic progression, and, in general, for all $n \geq 1$, the terms $a_{2n-1}, a_{2n}, anda_{2n+1}$ are in geometric progression, and the terms $a_{2n}, a_{2n+1}, anda_{2n+2}$ are in arithmetic progression. Let an be the greatest term in this sequence that is less than 1000. Find $n + a_n$.

Problem 13. (AIME, 2005) Let m be a positive integer, and let $a_0, a_1, ..., a_m$ be a sequence of real numbers such that $a_0 = 37$, $a_1 = 72, a_m = 0$, and

$$a_{k+1} = a_{k-1} - \frac{3}{a_k}$$

for $k = 1, 2, ..., m - 1$. Find m.

Problem 14. (USAMO, 2003) Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all real number x and y.